RANGE, DOMAIN OF TRIGONOMETRIC & INVERSE TRIGONOMETRIC FUNCTIONS

2.2 Basic Concepts

In Class XI, we have studied trigonometric functions, which are defined as follows:

sine function, i.e., sine : $\mathbf{R} \rightarrow [-1, 1]$

cosine function, i.e., cos : $\mathbf{R} \rightarrow [-1, 1]$

tangent function, i.e., tan : $\mathbf{R} - \{x : x = (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\} \rightarrow \mathbf{R}$

cotangent function, i.e., cot : $\mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\} \rightarrow \mathbf{R}$

secant function, i.e., sec : $\mathbf{R} - \{x : x = (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\} \rightarrow \mathbf{R} - (-1, 1)$

cosecant function, i.e., cosec : $\mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\} \rightarrow \mathbf{R} - (-1, 1)$

Since the domain of sine function is the set of all real numbers and range is the closed interval $[-1, 1]$. If we restrict its domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then it becomes one-one and onto with range $[-1, 1]$. Actually, sine function restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc., is one-one and its range is $[-1, 1]$. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$ (arc sine function). Thus, $\sin^{-1}$ is a function whose domain is $[-1, 1]$ and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and so on. Corresponding to each such interval, we get a *branch* of the function $\sin^{-1}$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the *principal value branch, whereas other intervals as range give different branches of* $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is $[-1, 1]$ and range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. We write $\sin^{-1} : [-1, 1] \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

From the definition of the inverse functions, it follows that $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. In other words, if $y = \sin^{-1} x$, then $\sin y = x$.

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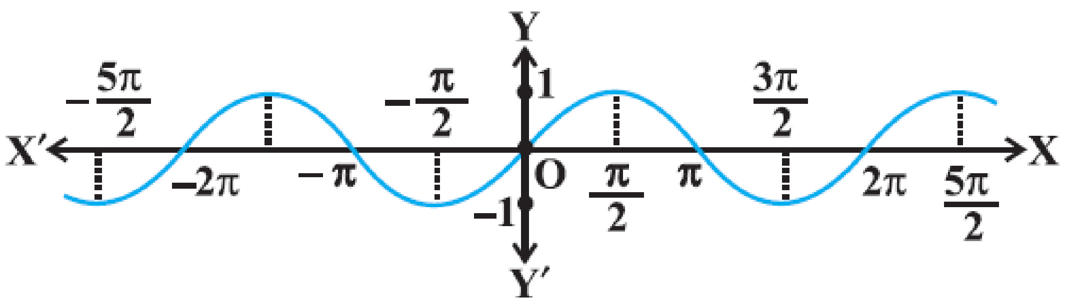
Remarks

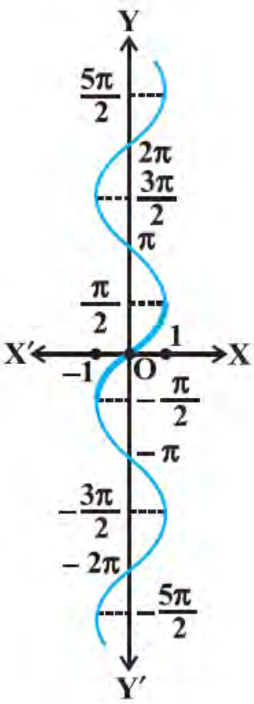
(i) We know from Chapter 1, that if $y = f(x)$ is an invertible function, then $x = f^{-1}(y)$. Thus, the graph of $\sin^{-1}$ function can be obtained from the graph of original function by interchanging x and y axes, i.e., if (a, b) is a point on the graph of sine function, then (b, a) becomes the corresponding point on the graph of inverse of sine function. Thus, the graph of the function $y = \sin^{-1} x$ can be obtained from the graph of $y = \sin x$ by interchanging x and y axes. The graphs of $y = \sin x$ and $y = \sin^{-1} x$ are as given in Fig 2.1 (i), (ii), (iii). The dark portion of the graph of $y = \sin^{-1} x$ represent the principal value branch.

(ii) It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line $y = x$. This can be visualised by looking the graphs of $y = \sin x$ and $y = \sin^{-1} x$ as given in the same axes (Fig 2.1 (iii)).



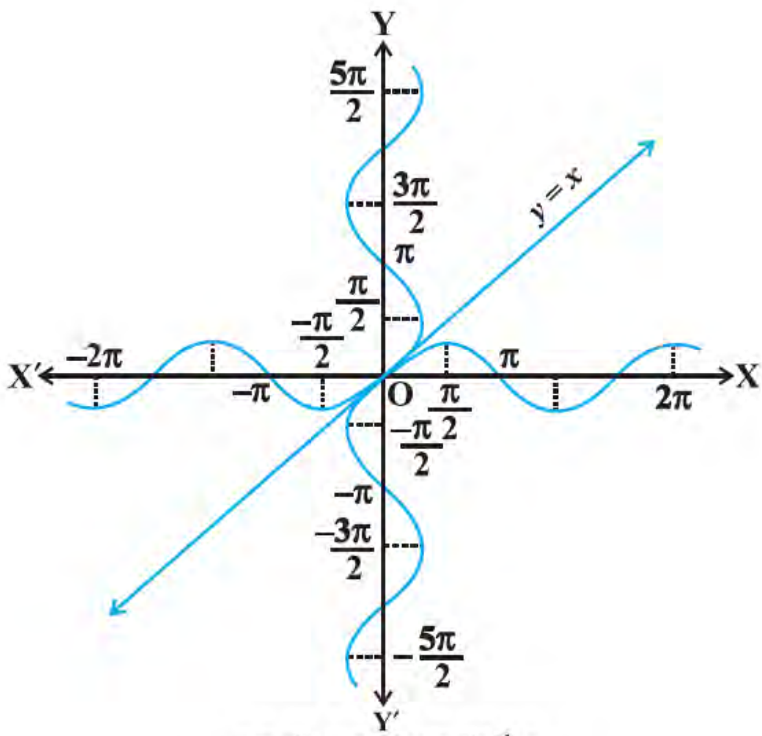
$y = \sin x$

Fig 2.1 (i)



$y = \sin^{-1} x$

Fig 2.1 (ii)



$y = \sin x$ and $y = \sin^{-1} x$

Fig 2.1 (iii)

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Like sine function, the cosine function is a function whose domain is the set of all real numbers and range is the set [–1, 1]. If we restrict the domain of cosine function to [0, π], then it becomes one-one and onto with range [–1, 1]. Actually, cosine function

restricted to any of the intervals [– π, 0], [0,π], [π, 2π] etc., is bijective with range as [–1, 1]. We can, therefore, define the inverse of cosine function in each of these intervals. We denote the inverse of the cosine function by $\cos^{-1}$ (arc cosine function).

Thus, $\cos^{-1}$ is a function whose domain is [–1, 1] and range could be any of the intervals [–π, 0], [0, π], [π, 2π] etc. Corresponding to each such interval, we get a branch of the function $\cos^{-1}$. The branch with range [0, π] is called the *principal value branch* of the function $\cos^{-1}$. We write

$$\cos^{-1} : [-1, 1] \to [0, \pi].$$

The graph of the function given by $y = \cos^{-1} x$ can be drawn in the same way as discussed about the graph of $y = \sin^{-1} x$. The graphs of $y = \cos x$ and $y = \cos^{-1} x$ are given in Fig 2.2 (i) and (ii).

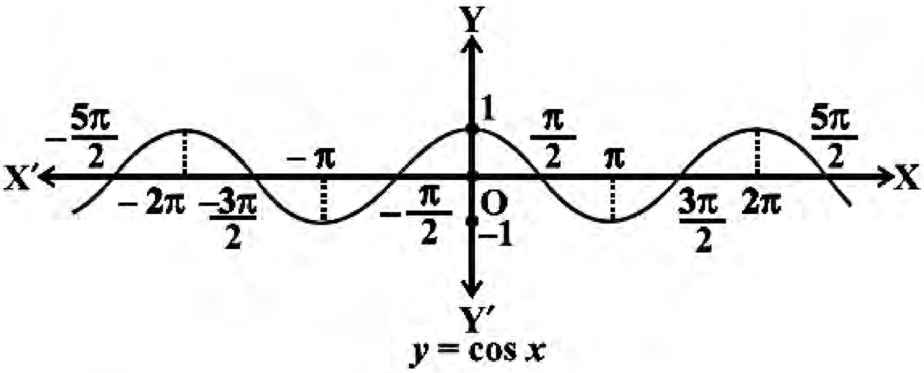

Fig 2.2 (i)

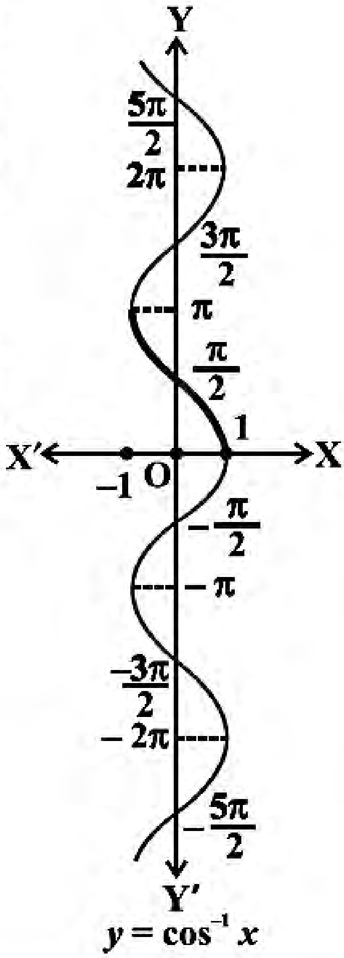

Fig 2.2 (ii)

Let us now discuss $\text{cosec}^{-1}x$ and $\sec^{-1}x$ as follows:

Since, $\text{cosec}\, x = \frac{1}{\sin x}$, the domain of the cosec function is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi,\ n \in \mathbf{Z}\}$ and the range is the set $\{y : y \in \mathbf{R},\ y \geq 1$ or $y \leq -1\}$ i.e., the set $\mathbf{R} - (-1, 1)$. It means that $y = \text{cosec}\, x$ assumes all real values except $-1 < y < 1$ and is not defined for integral multiple of π. If we restrict the domain of cosec function to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, then it is one to one and onto with its range as the set $\mathbf{R} - (-1, 1)$. Actually, cosec function restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{-\pi\}$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$ etc., is bijective and its range is the set of all real numbers $\mathbf{R} - (-1, 1)$.

Thus cosec^{-1} can be defined as a function whose domain is $\mathbf{R} - (-1, 1)$ and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{-\pi\}$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$ etc. The function corresponding to the range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is called the *principal value branch* of cosec^{-1}. We thus have principal branch as

$$\text{cosec}^{-1} : \mathbf{R} - (-1, 1) \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

The graphs of $y = \text{cosec}\, x$ and $y = \text{cosec}^{-1} x$ are given in Fig 2.3 (i), (ii).

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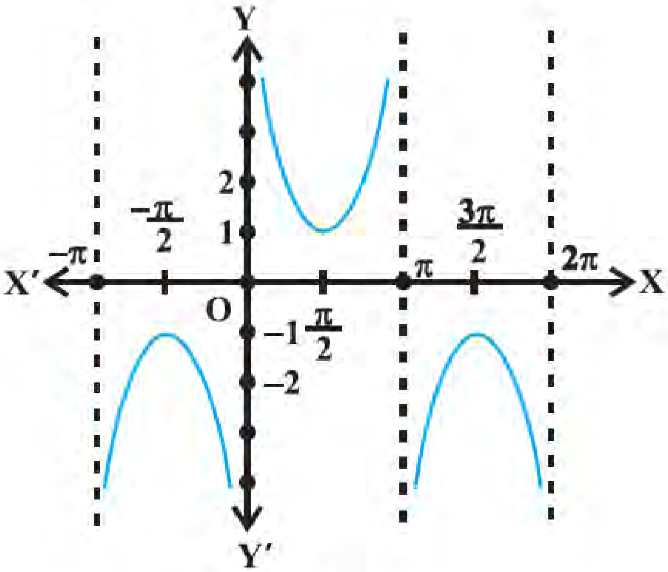
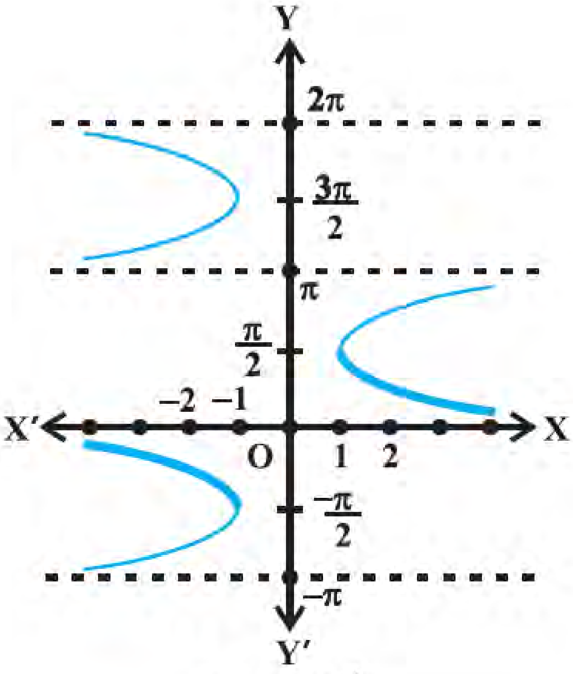


$y = \text{cosec}\, x$

Fig 2.3 (i)



$y = \text{cosec}^{-1} x$

Fig 2.3 (ii)

Also, since $\sec x = \frac{1}{\cos x}$, the domain of $y = \sec x$ is the set $\mathbf{R} - \{x : x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}\}$ and range is the set $\mathbf{R} - (-1, 1)$. It means that sec (secant function) assumes all real values except $-1 < y < 1$ and is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of secant function to $[0, \pi] - \{\frac{\pi}{2}\}$, then it is one-one and onto with its range as the set **R** – (–1, 1). Actually, secant function restricted to any of the intervals $[-\pi, 0]-\{\frac{-\pi}{2}\}$, $[0, \pi]-\left\{\frac{\pi}{2}\right\}$, $[\pi, 2\pi]-\{\frac{3\pi}{2}\}$ etc., is bijective and its range is $\mathbf{R}-\{-1, 1\}$. Thus $\sec^{-1}$ can be defined as a function whose domain is **R**– (–1, 1) and range could be any of the intervals $[-\pi, 0]-\{\frac{-\pi}{2}\}$, $[0, \pi]-\{\frac{\pi}{2}\}$, $[\pi, 2\pi]-\{\frac{3\pi}{2}\}$ etc. Corresponding to each of these intervals, we get different branches of the function $\sec^{-1}$. The branch with range $[0, \pi]-\{\frac{\pi}{2}\}$ is called the *principal value branch* of the function $\sec^{-1}$. We thus have

$$\sec^{-1} : \mathbf{R}-(-1,1) \rightarrow [0, \pi]-\{\frac{\pi}{2}\}$$

The graphs of the functions $y = \sec x$ and $y = \sec^{-1} x$ are given in Fig 2.4 (i), (ii).

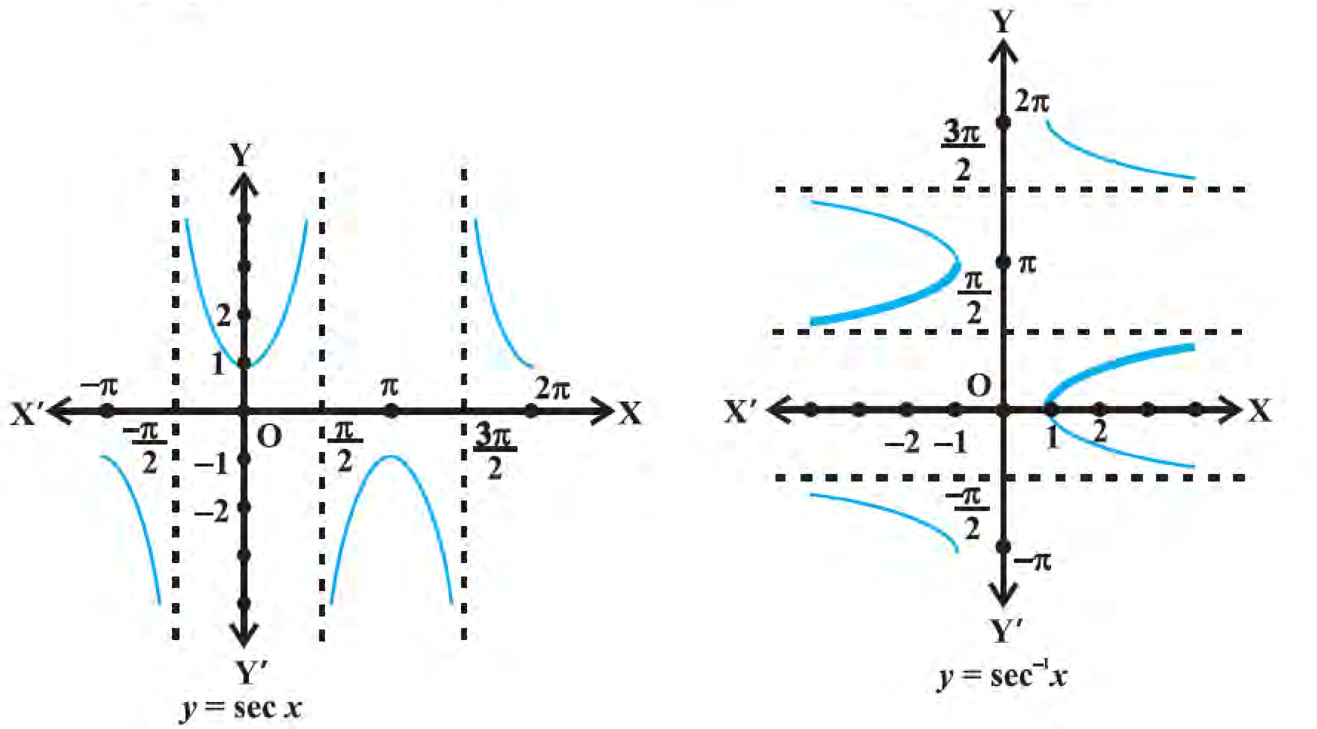


Fig 2.4 (i) Fig 2.4 (ii)

Finally, we now discuss $\tan^{-1}$ and $\cot^{-1}$

We know that the domain of the tan function (tangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and the range is **R**. It means that tan function is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of tangent function to

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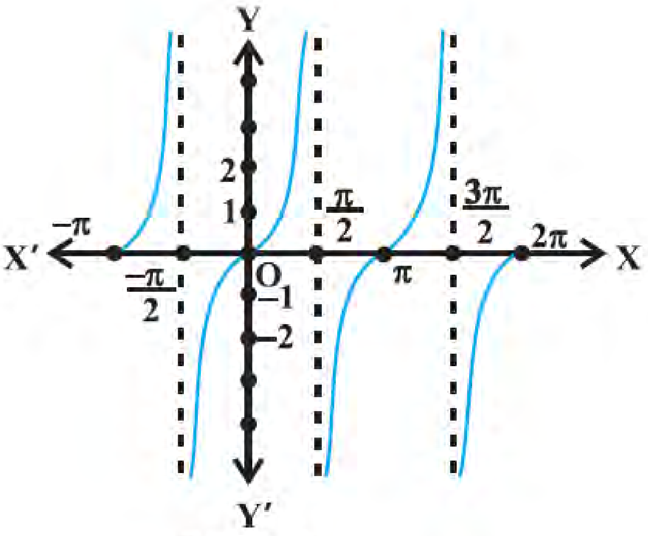
$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then it is one-one and onto with its range as **R**. Actually, tangent function restricted to any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ etc., is bijective and its range is **R**. Thus $\tan^{-1}$ can be defined as a function whose domain is **R** and range could be any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and so on. These intervals give different branches of the function $\tan^{-1}$. The branch with range $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is called the *principal value branch* of the function $\tan^{-1}$.

We thus have

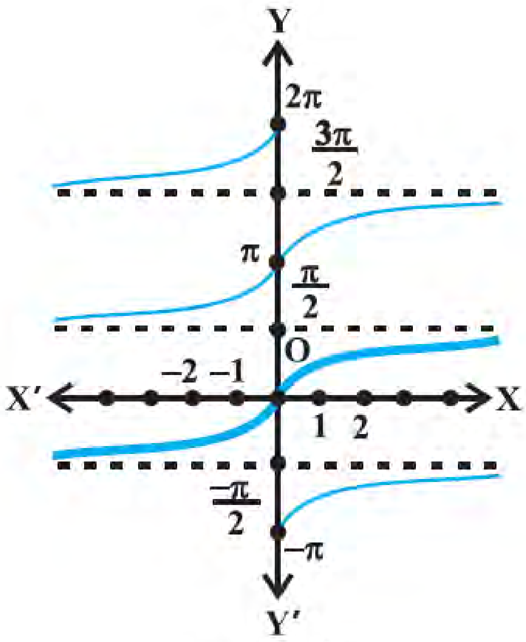
$$\tan^{-1} : \mathbf{R} \rightarrow \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of the function $y = \tan x$ and $y = \tan^{-1}x$ are given in Fig 2.5 (i), (ii).



$y = \tan x$

Fig 2.5 (i)



$y = \tan^{-1}x$

Fig 2.5 (ii)

We know that domain of the cot function (cotangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq n\pi, n \in \mathbf{Z}\}$ and range is **R**. It means that cotangent function is not defined for integral multiples of π. If we restrict the domain of cotangent function to $(0, \pi)$, then it is bijective with and its range as **R**. In fact, cotangent function restricted to any of the intervals $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$ etc., is bijective and its range is **R**. Thus $\cot^{-1}$ can be defined as a function whose domain is the **R** and range as any of the

intervals $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$ etc. These intervals give different branches of the function $\cot^{-1}$. The function with range $(0, \pi)$ is called the *principal value branch* of the function $\cot^{-1}$. We thus have

$$\cot^{-1} : \mathbf{R} \to (0, \pi)$$

The graphs of $y = \cot x$ and $y = \cot^{-1}x$ are given in Fig 2.6 (i), (ii).

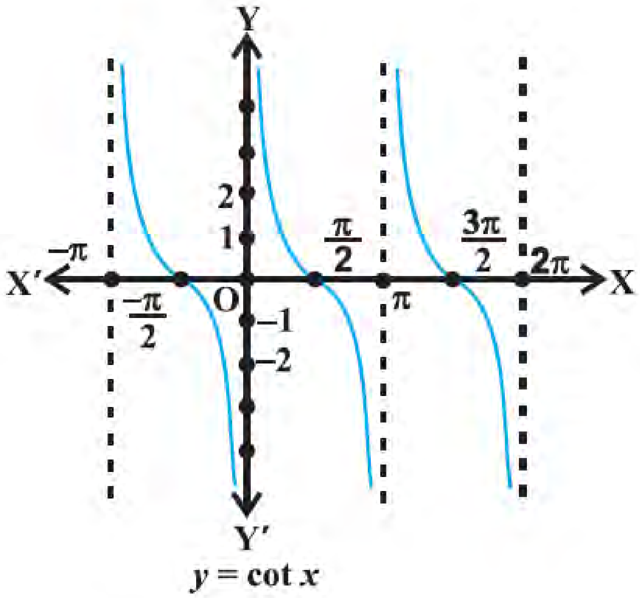


Fig 2.6 (i)

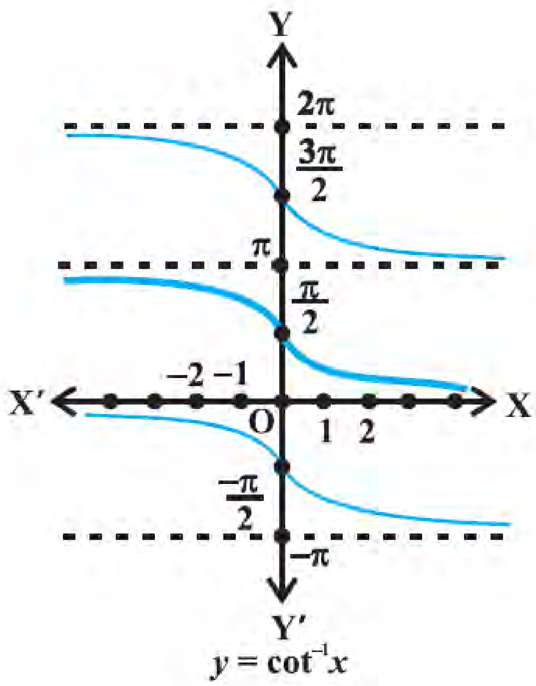


Fig 2.6 (ii)

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The following table gives the inverse trigonometric function (principal value branches) along with their domains and ranges.

$\sin^{-1}$	:	$[-1, 1]$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	:	$[-1, 1]$	$\rightarrow$	$[0, \pi]$
cosec^{-1}	:	$\mathbf{R} - (-1,1)$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	:	$\mathbf{R} - (-1, 1)$	$\rightarrow$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\tan^{-1}$	:	$\mathbf{R}$	$\rightarrow$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	:	$\mathbf{R}$	$\rightarrow$	$(0, \pi)$

Note

1. $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
2. Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.
3. The value of an inverse trigonometric functions which lies in the range of principal branch is called the *principal value* of that inverse trigonometric functions.